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Boundary kinematic control of a distributed oscillatory system ☆

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Abstract

The problem of the boundary kinematic control of one-dimensional oscillatory systems with distributed parameters in a finite time interval is investigated. The displacement of one of the ends is assumed to be controlled; the other end is assumed to be free. The system is subjected to additional disturbing actions, distributed and concentrated at the ends. The problem is stated by selecting an admissible control to transfer the system from an arbitrary state to the required final state, provided that the integral quadratic functional is minimal. Using the Fourier method, the maximum principle method and the *L*-problem of moments method, the problem of optimal control is solved explicitly in closed form, and, by d'Alembert's wave propagation method, is represented in terms of the initial functions. The requirements concerning the disturbing actions are established, and also those concerning the initial and final distributions and the duration of the control process, leading to smooth solutions of the problem. The problem of synthesizing the optimal control is discussed. The results are of interest when investigating problems of precision control of mechanical systems possessing considerable elastic compliance (shafts, beams, springs, strings, etc.)

1. Formulation of the problem

Consider a one-dimensional oscillatory system, with distributed parameters, describing the motions of elastic mechanical systems such as shafts, beams, distributed springs, strings, etc.[1,2] It is assumed that control and disturbing actions, both dynamic (forces or moments of forces) and kinematic (displacements, velocities or accelerations) (see Refs 3–9, etc.), can be applied to points (cross-sections).

The solution of problems of control and optimization in a finite time interval by actions concentrated at boundary points is of considerable theoretical and practical interest. For this purpose, methods of functional analysis, optimal control theory and the *L*-problem of moments are used, as well as approximate analytical and numerical methods.[1,2,10–14] Using these methods, solutions of specific problems have been obtained in fairly complete form.[3–5,7–9] They indicate the qualitative differences in controlled systems with distributed and lumped parameters, related to the finite propagation velocity of the control actions. In the general case of the initial distributions of displacements and velocities of points of the system, this property imposes a lower limit on the length of the time interval in which the control is being carried out. The properties of the controlled oscillators are considerably influenced by the different types of boundary conditions and control actions, and also the different types of initial and final distribution of the displacements and velocities.

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We will investigate the case of a "rigid" kinematic control concentrated at the left-hand end ($x = 0$) of the system. To fix our ideas, we will consider the governing equation and boundary conditions of the form

$$\rho\ddot{u} = \sigma u'' + U(t, x), \quad 0 < t < T < \infty, \quad 0 < x < l < \infty \tag{1.1}$$

$$u(t, 0) = a + A(t), \quad \sigma u'(t, l) = M(t) \tag{1.2}$$

where $u = u(t,x)$ is the angular or linear displacement of points of the system, ρ is the linear inertial characteristic, σ is the torsional stiffness (shafts) or the stiffness with respect to elastic displacement (beams or springs) or cross-sectional stiffness (taut threads or strings), l is the length and T is the time at which the control process ends. The functions U, A and M are considered to be specified and fairly smooth. The quantity a is the control (the rotation or displacement of the left-hand end) and is selected depending on the task in hand, namely to transfer the system from the known initial state at $t = 0$ to the required final state at $t = T$

$$u(0, x) = f^0(x), \quad \dot{u}(0, x) = g^0(x) \tag{1.3}$$

$$u(T, x) = f^T(x), \quad \dot{u}(T, x) = g^T(x) \tag{1.4}$$

Here, the integral quadratic functional of the quality taken as the performance index of the controlled motion should reach a minimum:

$$J[a] = \int_0^T a^2 dt \to \min_a, \quad |a| < \infty \tag{1.5}$$

Note that the values of the functions $f^{0,T}$ and $g^{0,T}$ and their derivatives at $x = 0, l$ should be consistent with the known values of A and M at $t = 0, T$ (see below). A similar problem was solved in Ref. 8 for $\dot{u}(t, 0) = a$, in Ref. 7 for the case of force control $u'(t, 0) = -a$, while in Ref. 9 a control concentrated at the ends and continuously distributed at points of the system was investigated.

An investigation of the initial problem of control and optimization using relations (1.1) to (1.5) is extremely tedious. It can be simplified considerably by using solution $z(t,x)$ of the "initial–boundary–value" problem

$$\begin{aligned} &\rho\ddot{z} = \sigma z'' + U(t, x), \quad z(t, 0) = A(t), \quad \sigma'(t, l) = M(t) \\ &z(T, x) = f^T(x), \quad \dot{z}(T, x) = g^T(x), \quad 0 < x < l, \quad T > t > 0 \end{aligned} \tag{1.6}$$

Note that here the "initial" data are taken at $t = T$, and the solution z is considered for $t < T$. It can be constructed in the form of a Fourier series using the standard procedure of the method of separation of variables (see below). It is assumed that the functions U, A, M, f^T and g^T belong to the class of fairly smooth (according to Steklov) functions.[10,11]

This requirement is governed by the properties of smoothness of the solution (classical, strong or weak[3,6,11]). The classical and, in part, strong solutions are of practical interest.

Suppose the classical solution of problem (1.6) is constructed in the form of a set of Fourier series, defined by the functions indicated. Each series can be "contracted" by d'Alembert's wave propagation method into a finite expression,[5,7–9] which is of fundamental importance for applications. Then, using the charge of variables $u = z + v$ and proceeding to dimensionless quantities, the initial problem of optimal control equations, (1.1)–(1.5), can be reduced, for the unknowns $a(t)$ and $v(t, x)$, to the following form

$$\begin{aligned} &\ddot{v} = v'', \quad 0 < t < T, \quad 0 < x < 1; \quad v(t, 0) = a, \quad v'(t, 1) = 0 \\ &v(0, x) = f(x) \equiv f^0(x) - z(0, x), \quad \dot{v}(0, x) = g(x) \equiv g^0(x) - z'(0, x) \\ &v(T, x) = \dot{v}(T, x) \equiv 0; \quad J[a] \to \min_a \end{aligned} \tag{1.7}$$

We take as the unit of length and time, l and $l(\rho/\sigma)^{1/2}$, respectively.

According to relations (1.7), a distributed system is considered, the right-hand end ($x = 1$) of which is free, while the left-hand end ($x = 0$) is subjected to a "rigid" control of the kinematic type. It is required to transfer the system as a whole optimally into a state of rest; when $a \equiv 0$ for $t \geq T$ it will remain in this state. Note that the initial data $f(x)$ and $g(x)$ for v and $\dot{v}$ respectively are determined using the known functions f^0 and g^0 and the constructed solutions z and

$\dot{z}$ at $t=0$ (see system (1.7)). They are assumed to be fairly smooth in the sense defined below when constructing the optimal motion of the system.

2. The use of the method of separation of variables and of the maximum principle

We will construct the required solution of the initial-boundary-value problem (1.7) with some fixed control $a(t)$ using the standard procedure of separation of variables: $v(t, x) \sim \theta(t)X(x)$. The coordinate function X and the constant of separation λ^2 are defined simply as the solution of the problem for eigenvalues and functions of the form

$$X'' + \lambda^2 X = 0, \quad X(0) = X'(1) = 0; \quad X_n(x) = \sin\lambda_n x$$
$$\lambda_n = \left(n - \frac{1}{2}\right)\pi, \quad \|X_n\| = \frac{1}{\sqrt{2}}, \quad n = 1, 2, \ldots \tag{2.1}$$

The system of functions X_n is orthogonal and complete. On its basis, the function $v(t, x)$ is constructed in the form of a Fourier series[1,2,14]

$$v(t, x) = \sum_{m=1}^{\infty} \theta_m(t)\sin\lambda_m x \tag{2.2}$$

Expression (2.2) becomes identically zero at $x=0$, i.e. a singularity occurs that disappears for $x>0$ $(x=+0)$. This property follows from subsequent constructions as a result of determining the unknown Fourier coefficients $\theta_n(t)$.

We will use Grinberg's method[10] is calculate the unknowns θ_n without reduction to the form of the problem with homogeneous boundary conditions. We differentiate expression (2.2) twice with respect to t and substitute Eqs. (1.7) into the left-hand side; the right-hand side remains unchanged. We now multiply both sides of the relation by $\sin\lambda_n x$ and integrate with respect to x in the interval $0 \le x \le 1$. This enables us to obtain a denumerable system of inhomogeneous differential equations for θ_n; the initial data are found in the standard way.[8] As a result, a non-trivial denumerable-dimensional problem of optimal control is obtained, containing a scalar control function a,

$$\ddot{\theta}_n + \lambda_n^2\theta_n = 2\lambda_n a, \quad n = 1, 2, \ldots, \quad 0 < t < T$$
$$\theta_n(0) = f_n \; : A = 10^3 \text{ кг}\cdot\text{м}^2, \quad B = C = 10^4 \text{ кг}\cdot\text{м}^2, \quad \Delta = 2 \text{ м} \; \int_0^1 (x)\sin\lambda_n x dx \tag{2.3}$$
$$\theta_n(T) = \dot{\theta}_n(T) = 0; \quad J[a] \to \min_a, \quad |a| < \infty$$

Provided that the function $a(t)$ is known, solutions of the Cauchy problem (2.3) are easily found and have the form

$$\theta_n(t) = f_n c_n(t) + \frac{g_n}{\lambda_n}s_n(t) + 2\int_0^t a(\tau)s_n(t-\tau)d\tau$$
$$\dot{\theta}_n = \frac{d\theta_n}{dt}, \quad c_n \equiv \cos\lambda_n t, \quad s_n \equiv \sin\lambda_n t \tag{2.4}$$

After substituting of expressions (2.4) into series (2.2) and some trigonometric transformations, the required function $v(t, x)$ can be represented, using d'Alembert's ("leftward" and "rightward") wave propagation method, by the expression

$$v(t,x) = \frac{1}{2}F(x-t) + \frac{1}{2}F(x+t) + \frac{1}{2}\int_0^t [G(x-\tau) + G(x+\tau)]d\tau +$$

$$+\int_0^t [\Delta(x-(t-\tau)) - \Delta(x+(t-\tau))]a(\tau)d\tau$$

$$F(y) = \sum_{n=1}^{\infty} f_n s_n(y), \quad G(y) = \sum_{n=1}^{\infty} g_n s_n(y) \tag{2.5}$$

$$F = \begin{cases} f(y), & 0 \le y \le 1 \\ f(2-y), & 1 < y \le 2 \end{cases}, \quad G = \begin{cases} g(y), & 0 \le y \le 1 \\ g(2-y), & 1 < y \le 2 \end{cases}$$

$$\Delta(y) = \sum_{n=1}^{\infty} c_n(y), \quad F(y) = -F(y-2), \quad G(y) = -G(y-2), \quad 2 < y \le 4$$

The functions F, G and Δ are periodic, with a period of 4, which follows from the property of 4-periodicity of the functions $s_n(y)$ and $c_n(y)$. Here, F and G are odd functions, and Δ is an even function (with respect to zero).

The correctness of expressions (2.2)–(2.4) can be confirmed by a change of variable leading to a zero boundary condition at $x = 0$. In fact, suppose $\ddot{a}$ is integrable and $\dot{a}$ is continuous. Then, by means of the obvious change of variable $v \to w$

$$v = h(x)a + w, \quad h'(1) = 0, \quad h(0) = 1 \tag{2.6}$$

for the unknown w, an initial–boundary–value problem is obtained that contains an inhomogeneity in the governing equation:

$$\begin{aligned} \ddot{w} &= w'' + h''(x)a - h(x)\ddot{a}, \quad w(t,0) = w'(t,1) = 0 \\ w(0,x) &= f(x) - h(x)a(0), \quad \dot{w}(0,x) = g(x) - h(x)\dot{a}(0) \end{aligned} \tag{2.7}$$

In expressions (2.6) and (2.7), the function $h(x)$ is twice continuously differentiable, and is arbitrary in all other respects; in particular, $h = (1 - x)^2$.

The use of the procedure of separation of variables leads to expression of the function $w(t, x)$ in the form of a Fourier series similar to (2.2). The inhomogeneity in Eq. (2.7) and the initial data are exposed in terms of a complete orthogonal system of functions X_n (2.1). The corresponding expressions for the Fourier coefficients (of the type θ_n and $\dot{\theta}_n$ (2.4)), after integration by parts and substitution into the series, yield identical representations for the required function $v(t, x)$.

Note that the function $\Delta(y)$ is a distribution generalized (when function); $y = 0, \pm 2, \pm 4, \ldots$ series (2.5) diverges; it has the meaning of a periodic Dirac δ-function, taking into account the effect of the control action $a(t)$.[14]

The structure of the optimal control is defined using the maximum principle.[12] Formally, the denumerable-dimensional vectors φ_n and ψ_n, adjoint to θ_n and $\dot{\theta}_n = \nu_n$, respectively, are introduced, and the necessary conditions of optimality are written down:

$$\begin{aligned} &\sum_{n=1}^{\infty} [\nu_n\varphi_n + (2\lambda_n a - \lambda_n^2\theta_n)\psi_n] - a^2 \to \max_a \\ &a^* = \sum_{n=1}^{\infty} \lambda_n\psi_n, \quad \ddot{\psi}_n + \lambda_n^2\psi_n = 0, \quad n = 1, 2, \ldots \end{aligned} \tag{2.8}$$

The equations for ψ_n are integrated simply; as a result, the required representation of the optimum control a^* (2.8) as a 4-periodic function of time t, odd with respect to $t = \pm 2$, is obtained; we have

$$a^*(t) = \sum_{n=1}^{\infty} [\kappa_n c_n(t) + \chi_n s_n(t)]; \quad a^*(t+4) \equiv a^*(t), \quad a^*(t \pm 2) = -a^*(t) \tag{2.9}$$

The constants κ_n and χ_n are to be determined; after substituting expression (2.9) into formulae (2.4), and taking into account the final conditions (when $t = T$) for θ_n and $\dot{\theta}_n$, a denumerable-dimensional linear system is obtained. Its effective solution for arbitrary $T < \infty$ and functions f and g from a certain class of smoothness encounters considerable difficulty and is the main content of the investigation conducted below.

3. Construction of the optimal control

We substitute the expression $a^*(t)$ (2.9) into the denumerable system (2.4), taken at $t = T$, and equate the functions $\theta_n(T)$ and $\dot{\theta}_n(T)$ to zero according to the final conditions (2.3). Solving it, we represent the equations defining $a^*(t)$ in the equivalent form of moment relations[13,14]

$$\int_0^T a^*(t) s_n(t) dt = \frac{1}{2} f_n, \quad \int_0^T a^*(t) c_n(t) dt = -\frac{1}{2} \frac{g_n}{\lambda_n} \tag{3.1}$$

Note that the integrands are 2-periodic owing to the properties, established by relations (2.9), of the expression $a^*(t)$. Below, the system of trigonometric functions $s_n(t)$ and $c_n(t)$ is complete and orthonormalized in the interval $0 \le t \le 2$.[15]

Relations (3.1) and expression (2.9) specify the L-problem of moments,[5,13,14] the solution of which defines the required optimal control and motion of system (1.7). The solvability of the problm, the uniqueness of the solution and its explicit construction require the development of special approaches,[5,7–9] which is non-trivial. They are based on the possibility of representing quadratures (3.1) in the interval of periodicity $0 \le t \le 2$. In fact, we will represent the quantity $T = 2N + \Theta$ uniquely, where $N = 0, 1, 2, \ldots$, while Θ is the 'remainder', $0 \le \Theta < 2$; we then have[5,7–9]

$$\int_0^{2N+\Theta} a^*(t) s_n(t) dy = \int_0^2 A(t) s_n(t) dt, \quad \int_0^{2N+\Theta} a^*(t) c_n(t) dy = \int_0^2 A(t) c_n(t) dt \quad (3.2)$$
$$A(t) = (N+1) a^*(t), \quad 0 < t \le \Theta; \quad A(t) = N a^*(t), \quad \Theta < t \le 2 \tag{3.2}$$

The right-hand sides of the Eqs. (3.1), taking into account relations (3.2), comprise the Fourier coefficients of the unknown function $A(t)$ in the interval $0 \le t \le 2$. The required solution uniquely follows from relations (3.2) and (3.1), with $N \ge 1$, in the indicated time interval:

$$a^*(t) = \begin{cases} (N+1)^{-1} A(t), & 0 < t \le \Theta < 2 \\ N^{-1} A(t), & \Theta < t \le 2 \end{cases}$$
$$A(t) = \frac{1}{2} \sum_{n=1}^{\infty} \left[f_n s_n(t) - \frac{g_n}{\lambda_n} c_n(t) \right] = \frac{1}{2} F(t) + \frac{1}{2} \int_0^t G(\tau) d\tau - \frac{1}{2} \int_0^1 G(t) dt \tag{3.3}$$

If $N = 0$, we assume that $A(t) \equiv 0$ for $\Theta < t \le 2$; this condition imposes additional requirements on the properties of the functions $f(x)$ and $g(x)$.

According to relations (2.9), (3.1) and (3.3), the optimal control $a^*(t)$ in the interval $2 < t \le 4$ is continued in an odd form, while for $t > 4$ it is continued periodically. The optimal motion (quenching of the oscillations) of system (1.7) occurs according to expressions (2.5) after substituting $a^*(t)$ (3.3). However, the formulae obtained are extremely long and difficult to analyse.

We will consider the formulation of the problem for $T = 2N$, simplified with respect to the parameter T, which enables an N-fold change of the control action to be achieved as a result of the passage of a wave disturbance. In particular, in the shortest interval $T = 2$ ($N = 1$), sufficient for the unique solution of the L-problem of moments and the optimal control problem, from relations (3.1), (3.2) and (2.9) we obtain a denumerable set of conditions imposed on $a^*(t)$,

$$\int_0^2 a^*(t)s_n(t)dt = \frac{f_n}{2}, \quad \int_0^2 a^*(t)c_n(t)dt = -\frac{g_n}{2\lambda_n} \tag{3.4}$$

Thus, the Fourier coefficients (3.4) of the required function $a^*(t)$ in terms of the orthonormalized system $s_n(t)$, $c_n(t)$, $0 \le t \le 2$, enable easily verifiable expressions to be obtained in the form of series

$$a^*(t) = \frac{1}{2}\sum_{n=1}^{\infty}\left[f_n s_n(t) - \frac{g_n}{\lambda_n}c_n(t)\right] = \frac{1}{2}\sum_{n=1}^{\infty} f_n s_n(t) + \frac{1}{2}\sum_{n=1}^{\infty} g_n \int_1^t s_n(\tau)d\tau \tag{3.5}$$

The first series in the last link of the chain of equalities (3.5) will be represented in terms of the initial function $f(x)$, $0 \le x \le 1$ (see formulae (2.5)). In fact, for $0 \le t \le 1$, this property is obvious, while, for $1 < t \le 2$, the function $f(t)$ is continued symmetrically: the sum of the series is equal to $f(2 - t)$. As a result we have the explicit expression

$$\sum_{n=1}^{\infty} f_n s_n(t) = F(t), \quad F(t) \equiv \begin{cases} f(t), & 0 \le t \le 1 \\ f(2-t), & 1 < t \le 2 \end{cases} \tag{3.6}$$

The second series can be transformed in the following way

$$\begin{gathered} \sum_{n=1}^{\infty} g_n \int_0^t s_n(\tau)d\tau = \int_1^t \left(\sum_{n=1}^{\infty} g_n s_n(\tau)\right)d\tau = -\int_0^1 g(t)dt + \int_0^t G(\tau)d\tau \\ \sum_{n=1}^{\infty} g_n s_n(t) = G(t), \quad G(t) \equiv \begin{cases} g(t), & 0 \le t \le 1 \\ g(2-t), & 1 < t \le 2 \end{cases} \end{gathered} \tag{3.7}$$

From the properties of the functions $s_n(t)$ it follows that the functions $F(t)$ and $G(t)$ are continuously differentiable at the “point of joining” $t = 1$ if $f'(1) = 0$ and $g'(1) = 0$. In this case, at other points of the interval $0 \le x < 1$, the functions $f(x)$ and $g(x)$ are also considerd to be smooth. Thus, according to Eqs. (3.5)–(3.7), an explicit representation (in terms of the initial functions $f(x)$, $g(x)$, $0 \le x \le 1$) of the optimal control $a^*(t)$ in the interval $0 \le t \le T = 2$ is obtained:

$$a^*(t) = \frac{1}{2}F(t) + \frac{1}{2}\int_1^t G(\tau)d\tau \tag{3.8}$$

Substitution of the optimal control $a^*(t)$ (3.8) into the formula for $v(t, x)$ (2.5) leads to an expression for the solution that contains eight terms. They are defined in terms of the functions F and G and the control components corresponding to them. The terms describing the free motion and control can be paired up. This simplifies the verification of the property $v(T, x) = \dot{v}(T, x) \equiv 0$ at $T = 2$. If, for $t \ge T$, we assume the control $a(t) \equiv 0$, then the elastic system (1.7), in the absence of disturbances, will remain in a state of rest $v(t, x) \equiv 0$.

The solution constructed (weak, strong or classical) can be verified directly or by using a representation in the form of Fourier series. The properties of smoothness should be ensured by selection of the functions $f(x)$ and $g(x)$, i.e. the coefficients f_n and g_n should decrease rapidly as $n \to \infty$. In particular, it will be satisfied if the series (2.5) (for the interval $T = 2$) contain a finite number of terms. Then the expressions for the control a^* (3.8) and the motion v (2.5) will also contain only a finite number of terms.

Suppose, now, that $T = 2N$, where the integer $N \ge 2$; then, from the expression (3.3) it follows that the required value of the optimal control is defined in the form $a_N^*(t) = N^{-1}a*(t)$, where the function a^* is defined by expression (3.8), and its continuation occurs according to the formulae (2.5) and (2.9). From this it follows that, if the number N is fairly large, the optimal control $|a_N^*|$ can be as small as desired for specified $f(x)$ and $g(x)$. As a result, the geometric-type constraints normally imposed in applied research on the magnitude of the control action will be satisfied. Note that the value of the functional $J[a_N^*]$ approaches zero, as $N \to \infty$, as a quantity of the order N^{-1}, i.e., the integral constraints imposed on the control may also be satisfied by choosing the duration of the control process.

We will calculate the minimum value of the functional using relations (1.5) and (3.5):

$$J_1^* = J[a^*] = \frac{1}{4}\sum_{n=1}^{\infty}\left(f_n^2 + \frac{g_n^2}{\lambda_n^2}\right) = \frac{1}{2}\int_0^1\left[f^2(t) + \left(\int_1^t g(\tau)d\tau\right)^2\right]dt$$
$$J_N^* = J[a_N^*] = N^{-1}J_1^* = N^{-1}J[a^*] \qquad (3.9)$$

Thus, exact expressions for the optimal open-loop control and the optimal motion are constructed and represented in an explicit form in terms of the initial distribution functions. In the same way as expression (3.9), the value of the functional for the derivatives of T and the initial distributions $f(x)$ and $g(x)$ is calculated (taking into account the familiar constraint for $T < 2$ and the requirements of smoothness on f and g).

We will discuss the possibility of synthesizing the optimal feedback control. If considerable disturbances, ignored in the relations (1.1) and (1.2), are acting on the system, then, as they accumulate, the state of system (1.7) will change and it will be necessary, at certain times t (in theory, continuously), to take into account the actual distribution

$$f^t(x) = \hat{\upsilon}(t, x), \quad g^t(x) = \hat{\dot{\upsilon}}(t, x)$$

where $\hat{\upsilon}$ and $\hat{\dot{\upsilon}}$ are the measured state of the system, and also the time $T - t$ remaining until the end of the control process. This "synthesis" procedure can be implemented approximately, for example in discrete time intervals $\Delta t = 2$. At the final stage $T - t < 2$, the open-loop control $a^*(t)$ (3.3), or another method of minimizing the deviation from the required state of rest, is used.

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References

[1]. Timoshenko S. *Vibration Problems in Engineering*. New York–London: D. van Nostrand Co.; 1995.
[2]. Courant R, Hilbert D. *Methoden der Matematischen Physik, Bd 1*. Berlin: Springer; 1931.
[3]. Komkov V. *Optimal Control Theory for the Damping of Vibrations of Simple Elastic Systems*. Berlin: Springer; 1972.
[4]. Troitskii VA. *Optimal Processes of Vibrations of Mechanical Systems*. Leningrad: Mashinostroyeniye; 1976.
[5]. Butkovskii AG. *Methods for Control Systems with Distributed Parameters*. Moscow: Nauka; 1975.
[6]. Lions J-L. *Control Optimal de Systémes Gouvernés par des Equations aux Dérivées Partielles*. Paris: Dunod Gauthier-Villars; 1968.
[7]. Akulenko LD. The steering of an elastic system to a specified state by a force boundary action. *Prikl Mat Mekh* 1981;**45**(6):1095–103.
[8]. Akulenko LD, Bolotnik NN. Kinematic control of the motion of an elastic system. *Izv Akad Nauk SSSR MTT* 1984;(2):168–76.
[9]. Akulenko LD. Optimal control of simple motions of a homogeneous elastic body. *Izv Ross Akad Nauk MTT* 1992;(3):200–7.
[10]. Grinberg GA. A new method for solving some boundary–value problems for equations of mathematical physics allowing of the separation of variables. *Izv Akad Nauk SSSR Ser Fiz* 1946;**10**(2):141–68.
[11]. Steklov VA. *Fundamental Problems of Mathematical Physics*. Moscow: Nauka; 1983.
[12]. Pontryagin LS, Boltyanskii VG, Gamkrelidze RV, Mishchenko EF. *The Mathematical Theory of Optimal Processes*. New York: Wiley; 1969.
[13]. Krasovskii NN. *The Theory of Motion Control*. Moscow: Nauka; 1968.
[14]. Krein MG, Nudel'man AA. *Problem of Markov Moments and Extremal Problems*. Moscow: Nauka;; 1973.
[15]. Tolstov GP. *Fourier Series*. Moscow: Nauka; 1980.

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